

**A CLASS OF POWER ASSOCIATIVE LCC-LOOPS AND SOME
ASSOCIATED TOTAL INNER MAPPING GROUP QUESTIONS**†**

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Abstract

In this paper, we investigated the algebraic properties of a class of LCC-loops (and its dual) which is power associative, tagged ‘twist’ of LWPC (RWPC) viz. LTWC (RTWC). Its relationship with Frute and crazy loop was discovered. LTWC-loop was shown to be a class of loop which falls into the Syrbu and Grecu (2019) first answer to a question of Stanovský and Vojtěchovský (2014)-minimal generating set for the total inner mapping group of a loop. In addition, we showed that the class of LTWC-loop also provides an alternative solution to 2014 question and in fact, the inner automorphism group of a LTWC-loop is generated by just one inner mapping. Generalized forms of the left (right) alternative property and flexibility property were introduced and studied in this class of loop. We also explore the properties of the left and right translation of this class of loop and proved some results useful for the construction a finite loop belonging to this same class.

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1 Introduction

The LWPC and the RWPC loops were introduced by Phillips in [16]. George et al. [5] showed that each of these two loop identities described by

$$(xy \cdot x) \cdot xz = x((yx \cdot x)z) \quad (\text{LWPC})$$

$$zx \cdot (x \cdot yx) = (z(x \cdot xy))x \quad (\text{RWPC})$$

is power associative. They established that the following equations of identities are true in loops

$$(\text{LWPC}) = (\text{LCC}) + (\text{P}_\lambda) \text{ and } (\text{RWPC}) = (\text{RCC}) + (\text{P}_\rho), \text{ where}$$

$$\underbrace{(xy \cdot x)x = x(yx \cdot x)}_{\text{P}_\lambda} \quad \underbrace{x(x \cdot yx) = (x \cdot xy)x}_{\text{P}_\rho}$$

George and Jaíyéolá [6] investigated two identities (Q_{12} and Q_7) similar to LWPC and RWPC. These identities may be seen as ‘twist’ of LWPC and RWPC, a possible name for them might be LTWC and RTWC.

$$x((x \cdot yx)z) = (x \cdot xy) \cdot xz \quad (\text{LTWC})$$

$$(y(xz \cdot x))x = yx \cdot (zx \cdot x) \quad (\text{RTWC})$$

Similarly, they showed that the following equations of identities are true in loops

$$(\text{LTWC}) = (\text{LCC}) + (\text{P}_\rho) \text{ and } (\text{RTWC}) = (\text{RCC}) + (\text{P}_\lambda).$$

Likewise, since the equation of identities $(\text{LWPC}) + (\text{RWPC}) = (\text{CC}) + (\text{WIP})$ is true in loops, so also is $(\text{LTWC}) + (\text{RTWC}) = (\text{CC}) + (\text{WIP})$.

In [5], a method of the construction of LWPC-loop (with a necessary and sufficient condition) was developed (see Theorem 3.1 for details) with the aid of a loop (G, R, f) originally constructed in [3, 4] where R is a subgroup of an abelian group G and $f : G \times G \rightarrow R$ is a mapping such that $\text{Rad}(f) \geq R$. The semidirect product of the LWPC and RWPC loops were considered in [7].

Below are few generalized forms of identities of Bol-Moufang type of loops

which will be of interest in this current work. Note that some of them have equivalent forms of identities. See Cote et al. [19].

$$\begin{array}{ll}
 \text{Frute : } x(x \cdot yz) = yx \cdot xz & A1C3 \\
 \text{Frute : } x(xy \cdot z) = y(x \cdot xz) & A2C1 \\
 \text{Crazy : } (x(xy))z = (yx)(xz) & A4C3 \\
 \text{Crazy : } x(xy \cdot z) = (yx \cdot x)z & A2C5
 \end{array}$$

Frute loops have been studied in Jaiyéolá et al. [12, 14]. They are Moufang loops while a crazy loop implies any of the Bol-Moufang type loops including groups. A loop has been shown to be a Cheban loop if and only if it is both a left and right Cheban loop in [17].

The purpose of this work is to introduce and study generalized forms of the left (right) alternative property and flexibility properties in LTWC-loop and to show that the inner automorphism group of a LTWC-loop is generated by just one inner mapping.

2 Preliminaries

A loop Q is a set with a binary operation $\cdot$ such that for all $a, b \in Q$, the equations $ax = b$ and $ya = b$ have unique solutions x and y in Q and there exists $e \in Q$ such that $ex = xe = x$ for any $x \in Q$. The unique solutions are given by $x = a \setminus b$ and $y = b/a$. The reader can consult the books [2, 10, 15, 20, 24] for a general background in loop theory.

In a loop Q with identity e , for all $x \in Q$ there exists a unique right inverse element x^ρ and a unique left inverse element x^λ such that $xx^\rho = x^\lambda x = e$. Note that x^ρ is not always equal to x^λ in a loop, when they are equal we write $x^\rho = x^\lambda = x^{-1}$.

In a groupoid $(Q, \cdot)$, for any $x \in Q$, define the right translation (multiplication) map $R_x = R(x)$ and left translation (multiplication) map $L_x = L(x)$ of x by $yR(x) = y \cdot x = yx$ and $yL(x) = x \cdot y = xy$, respectively. Then, $(Q, \cdot)$ is called a quasigroup if the left and right translation maps are bijections. Since the translation maps are bijections in a quasigroup, then the inverse maps $R^{-1}(x)$ and $L^{-1}(x)$ exist and are thus defined by

$yR^{-1}(x) = y/x = xM_y^{-1}$ and $yL^{-1}(x) = x \setminus y = xM_y$. A quasigroup with an identity element is a loop.

A loop $(Q, \cdot)$ is power associative if subloops generated by every single elements are groups. This is easily seen to be equivalent to $x^{m+n} = x^m \cdot x^n$ for every $x \in Q$ and for all $m, n \in \mathbb{Z}$. A loop $(Q, \cdot)$ is said to be 3-power associative if $xx \cdot x = x \cdot xx$ for all $x \in Q$.

Definition 2.1. Let $(G, \cdot)$ be a loop. G is said to be a left alternative property loop (LAPL) if for all $x, y \in G$, $x \cdot xy = xx \cdot y$, a right alternative property loop (RAPL) if for all $x, y \in G$, $yx \cdot x = y \cdot xx$, and an alternative loop if it is both left and right alternative.

A loop $(G, \cdot)$ is called a flexible or an elastic loop if the flexibility or elasticity property $xy \cdot x = x \cdot yx$ holds for all $x, y \in G$.

$(G, \cdot)$ is said to have the left inverse property (LIP) if for all $x, y \in G$, $x^\lambda \cdot xy = y$, the right inverse property (RIP) if for all $x, y \in G$, $yx \cdot x^\rho = y$ and the inverse property if it has both left and right inverse properties.

A loop $(G, \cdot)$ is called a weak inverse property (WIP) loop if it satisfies any one of the equivalent identities $x(yx)^\rho = y^\rho$ and $(xy)^\lambda x = y^\lambda$.

Let $n \in \mathbb{Z}$ be fixed. A power associative loop $(G, \cdot)$ is said to have the left n -power alternative (LPAⁿ) property if for all $x, y \in G$, $\underbrace{x(\cdots(x(xy))\cdots)}_{n\text{-times}} = x^n y$. If a power associative loop $(G, \cdot)$ has the LPAⁿ-property for all $n \in \mathbb{Z}$, then it is called a left power alternative loop.

Let $n \in \mathbb{Z}$ be fixed. A power associative loop $(G, \cdot)$ is said to have the right n -power alternative (RPAⁿ) property if for all $x, y \in G$, $\underbrace{(\cdots((yx)x)\cdots)}_{n\text{-times}} x = yx^n$. If a power associative loop $(G, \cdot)$ has the RPAⁿ-property for all $n \in \mathbb{Z}$, then it is called a right power alternative loop.

A power associative loop is said to be power alternative loop if it is both left and right power alternative loop. Conjugacy closed loops (CC-loop) are loops satisfying the following two identities:

$$zy \cdot x = zx \cdot x \setminus (yx). \quad (RCC)$$

$$x \cdot yz = (xy)/x \cdot xz. \quad (LCC)$$

The group of all permutations on a quasigroup G is called the permutation group of G and denoted by $SYM(G)$. The group $\mathcal{M}(G, \cdot) = \langle \{R_x, L_x, : x \in G\} \rangle$ is called the multiplication group of $(G, \cdot)$ and $\mathcal{M}(G, \cdot) \leq SYM(G)$. The group of automorphisms in $(G, \cdot)$ is denoted by $AUM(G, \cdot)$.

If $e\alpha = e$ in a loop G such that $\alpha \in \mathcal{M}(G)$, then α is called an inner mapping and they form a group $\text{Inn}(G)$ called the inner mapping group. The right, left and middle inner mappings

$$R_{(x,y)} = R_x R_y R_{xy}^{-1}, L_{(x,y)} = L_x L_y L_{yx}^{-1} \text{ and } T_x = R_x L_x^{-1}$$

respectively generate the left inner mapping group $\text{Inn}_\lambda(G)$, right inner mapping group $\text{Inn}_\rho(G)$ and the middle inner mapping group $\text{Inn}_\mu(G)$. If

$$\text{Inn}_\lambda(G) \leq AUM(G), \text{Inn}_\rho(G) \leq AUM(G), \text{ and } \text{Inn}_\mu(G) \leq AUM(G),$$

then G is called a left A-loop (A_λ -loop), right A-loop (A_ρ -loop) and middle A-loop (A_μ -loop) respectively. It is well known that $\text{Inn}(G, \cdot) = \langle R_{(x,y)}, L_{(x,y)}, T_x \mid x, y \in G \rangle$, but this was later improved to be $\text{Inn}(G, \cdot) = \langle L_{(x,y)}, T_x \mid x, y \in G \rangle = \langle R_{(x,y)}, T_x \mid x, y \in G \rangle$ in Vojtěchovský [23]. If $\text{Inn}(G, \cdot) \leq AUM(G, \cdot)$, then $(G, \cdot)$ is called an automorphic loop (A-loop). Some works in the direction of structure of A-loops can be found in [11, 23].

The group $\mathcal{TM}(G, \cdot) = \langle \{R_x, L_x, M_x : x \in G\} \rangle$ is called the total multiplication group of $(G, \cdot)$ and an element $\alpha \in \mathcal{TM}(G, \cdot)$ such that $e\alpha = e$ is called a total inner mapping of $(G, \cdot)$ and they form a group $\text{TInn}(G, \cdot)$ called the total inner mapping group. It has been established in Stanovský and Vojtěchovský [21] that $\text{TInn}(G, \cdot) = \langle R_{(x,y)}, L_{(x,y)}, T_x, M_{(x,y)}, U_x \mid x, y \in G \rangle$, where $M_{(x,y)} = M_y M_x M_{y \setminus x}^{-1}$ and $U_x = M_x R_x^{-1}$. Also, Syrbu and Grecu [22] showed that $\text{TInn}(G, \cdot) = \langle R_{(x,y)}, L_{(x,y)}, T_x, P_{(x,y)}, V_x \mid x, y \in G \rangle$

where $P_{(x,y)} = M_x M_y L_x R_y^{-1}$, and $V_x = R_x M_x$. If $\text{TInn}(G, \cdot) \leq AUM(G, \cdot)$, then $(G, \cdot)$ is called a totally automorphic loop (TA-loop). Some other works in the direction of structure of TA-loops can be found in Jaiyéolá and Effiong [13], Kinyon [18]. As mentioned in [13], [18] raised a question (Question 1.1) of what other interesting varieties of loops can be characterized by specifying that some group of total inner mappings acts as automorphisms? The authors in [13] provided a partial answer to Question 1.1.

Furthermore, the authors in [21] raised a question (Question 1.2 in [13]) if the generating sets of $\text{TInn}(G)$ for a loop G are minimal, i.e, can any

of the five types of mappings be removed? The authors in [13] provided a ostensible answer to Question 1.2. Syrbu and Greu [22] were able to provide answers to Question 1.2 by showing that:

1. If Q is a power associative loop, then

$$\text{TInn}(Q) = \langle R_{(x,y)}, L_{(x,y)}, P_{(x,y)}, V_x \mid x, y \in Q \rangle.$$

2. If Q is a middle Bol loop, then $\text{TInn}(Q) = \langle R_{(x,y)}, P_{(x,y)}, V_x \mid x, y \in Q \rangle.$

In this work, we shall show that LTWC-loop is a class of loop which falls into the Syrbu and Greu's first answer to Question 1.2. In addition, the class of LTWC-loop also provides an alternative solution to Question 1.2. in fact, the inner automorphism group of a LTWC-loop is generated by just one inner mapping.

3 Main Results

3.1 RTWC-Loop and LTWC-Loop

In a LTWC (RTWC)-loop, the following are equivalent:

1. Flexibility.
2. RAP (LAP).
3. RIP (LIP).

Hence, any LTWC (RTWC)-loop which obeys any of them is an extra loop. Any LTWC (RTWC)-loop is a Moufang loop if and only if it is an extra loop. In a LTWC-loop G , $N_\mu(G) = N_\lambda(G)$. While in a RTWC-loop G , $N_\mu(G) = N_\rho(G)$.

These observations necessitate the introduction of the properties in Definition 3.1 for investigation in addition to the fact that it was shown in [8] that a LTWC-loop is power associative.

Definition 3.1. *A power associative loop G is said to obey*

1. *LAP_n-property if $x \cdot x^n y = x^{n+1} y$ where $n \in \mathbb{Z}$, $\forall x, y \in G$.*
2. *LAPⁿ-property if $x^n \cdot xy = x^{n+1} y$ where $n \in \mathbb{Z}$, $\forall x, y \in G$.*
3. *FLEXⁿ-property if $x^n \cdot yx = x^n y \cdot x$ where $n \in \mathbb{Z}$, $\forall x, y \in G$.*

4. $FLEX_n$ -property if $x \cdot yx^n = xy \cdot x^n$ where $n \in \mathbb{Z}$, $\forall x, y \in G$.
5. $FLEX_{(2,n+1)}$ -property if $x^2y \cdot x^n = x^2 \cdot yx^n$ where $n \in \mathbb{Z}$, $\forall x, y \in G$.
6. RAP^n -property if $yx \cdot x^n = yx^{n+1}$ where $n \in \mathbb{Z}$, $\forall x, y \in G$.
7. RAP_n -property if $yx^n \cdot x = yx^{n+1}$ where $n \in \mathbb{Z}$, $\forall x, y \in G$.

Remark 3.2. In any power associative loop: LIP is equivalent to the property LAP^{-1} or LAP_{-1} ; RIP is equivalent to the property RAP^{-1} or RAP_{-1} ,

Theorem 3.3. Let G be a LTWC (RTWC)-loop. G is a Frute loop if and only if G is a crazy loop.

Proof. Let G be a LTWC-loop and suppose G is also a crazy (A4C3) loop, i.e., $(x \cdot xy)z = yx \cdot xz$. Put $z \mapsto xz$ in the crazy identity, then we have

$$(x \cdot xy) \cdot xz = yx \cdot (x \cdot xz) = x((x \cdot yx)z) \Rightarrow x(xy \cdot z) = y(x \cdot xz)$$

which is A2C1. So, G is a Frute loop.

Conversely, let G be a LTWC-loop and suppose also that G is also a Frute (A2C1) loop. Put $y = y/x$ in the LTWC identity, we have $x(xy \cdot z) = (x(x(y/x))) \cdot xz$, and so since G is also Frute loop (A2C1); $y(x \cdot xz) = (x(x(y/x))) \cdot xz$. Thus, G is a crazy loop. □

Remark 3.4. Theorem 3.3(3.3) is nontrivial because by (Theorem 2.6 (2) of [14]), in a Frute loop, the property $x^{-1}yx = yx^{-1}$ (which is equivalent to the centrum square property in a diassociative loop) is satisfied but not generally true in Moufang, extra loop or group. Hence, it distinguishes a Frute loop from a Moufang, extra loop and group. In (Theorem 2.2 of [14]), this property is necessary and sufficiency for a group to be a Frute loop and vice versa. But may not be necessary and sufficiency for a Frute loop to be a crazy loop (which is a group and has the property itself).

Theorem 3.5. Let G be a LTWC-loop.

1. $x \cdot x^n y = x^n \cdot xy \forall x, y \in G, n \in \mathbb{Z}$.

2. G is left power alternative $\Leftrightarrow G$ has the LAP_n -property for all $n \in \mathbb{Z} \Leftrightarrow G$ has the LAP^n -property for all $n \in \mathbb{Z}$.
3. For any fixed $n \in \mathbb{Z}$, G has the LAP^n -property $\Leftrightarrow G$ has the LAP_n -property.
4. G has the $FLEX_n$, LAP^1 and $FLEX_{(2,n+1)}$ properties implies that G has the RAP^n property for any fixed $n \in \mathbb{Z} \setminus \{1\}$.
5. For any fixed $n \in \mathbb{Z}$, the LPA^{n+1} and LPA^n properties imply the LAP^n and LAP_n properties.
6. Left power alternative property implies the LAP^n and LAP_n properties for all $n \in \mathbb{Z}$.
7. LAP^n or LAP_n and LPA^n properties imply the LPA^{n+1} property for any fixed $n \in \mathbb{Z}$.

Proof. 1. We claim that $x \cdot x^{n-1}y = x^{n-1} \cdot xy$ for all $x, y \in G$, $n \in \mathbb{Z}$.

For $n = 1$, $xy = xy$ (trivial).

For $n = 2$, $x \cdot xy = x \cdot xy$.

For $n = 3$, $x \cdot x^2y = x^2 \cdot xy$.

We assume that $x \cdot x^{n-1}y = x^{n-1} \cdot xy$ for all $n \leq k$, that is $x \cdot x^{k-1}y = x^{k-1} \cdot xy$.

Put $y = x^{k-2}$ in LTWC identity to get: $x \cdot (x \cdot x^{k-2}x)z = (x \cdot xx^{k-2}) \cdot xz \Rightarrow x \cdot (xx^{k-1})z = (xx^{k-1}) \cdot xz \Rightarrow x \cdot x^kz = x^k \cdot xz \Rightarrow x \cdot x^{(k-1)+1}z = x^{(k-1)+1} \cdot xz$. Replace $k - 1$ by n to get $x \cdot x^{n+1}z = x^{n+1} \cdot xz$. So, $x \cdot x^n z = x^n \cdot xz \forall n \in \mathbb{N}$.

Put $y = x^{-k-2}$ in LTWC to get: $x \cdot (x \cdot x^{-k-2}x)z = (x \cdot xx^{-k-2}) \cdot xz \Rightarrow x \cdot (xx^{-k-1})z = (xx^{-k-1}) \cdot xz \Rightarrow x \cdot x^{-k}z = x^{-k} \cdot xz$. So, $x \cdot x^k y = x^k \cdot xy$, $\forall k \in \mathbb{Z}^-$. Therefore, $x \cdot x^n y = x^n \cdot xy \forall x, y \in G$, $n \in \mathbb{Z}$.

2. G is left power alternative $\Leftrightarrow \underbrace{x(\cdots(x(xy))\cdots)}_{n+1\text{-times}} = x^{n+1}y \ \forall x, y \in G$
and $n \in \mathbb{Z}$. With 1, it will be observed that:

$$\underbrace{x(\cdots(x(xy))\cdots)}_{n\text{-times}} = x^n y$$

implies $x \cdot x^{n-1}y = x^n y$ and $x^{n-1} \cdot xy = x^n y$ for all $x, y \in G$ and $n \in \mathbb{Z}$.
Likewise,

$$x \cdot x^{n-1}y = x^n y \text{ and } x^{n-1} \cdot xy = x^n y \text{ when } n = 1, 2, 3, \dots, i$$

imply

$$\underbrace{x(\cdots(x(xy))\cdots)}_{n\text{-times}} = x^n y \text{ when } n = i + 1$$

for all $x, y \in G$ and $i \in \mathbb{Z}$.

3. Use 1.

4. Put $z = x$ in LTWC to get $x[(x \cdot yx)x] = (x \cdot xy)x^2$. Then,

$$\begin{aligned} \text{FLEX}_1 \text{ property in } G &\Rightarrow x[x(yx \cdot x)] = (x \cdot xy)x^2 \xrightarrow[\text{property}]{\text{LAP}^1} \\ x^2(yx \cdot x) &= x^2y \cdot x^2 \xrightarrow[\text{property}]{\text{FLEX}_{(2,2)}} x^2(yx \cdot x) = x^2 \cdot yx^2 \Rightarrow \\ yx \cdot x &= yx^2 \Rightarrow \text{RAP}^1 \text{ property.} \end{aligned}$$

Put $z = x^2$ in LTWC to get $x[(x \cdot yx)x^2] = (x \cdot xy)x^3$. Then,

$$\begin{aligned} \text{FLEX}_2 \text{ property in } G &\Rightarrow x[x \cdot (yx)x^2] = (x \cdot xy)x^3 \xrightarrow[\text{property}]{\text{LAP}^1} \\ x^2(yx \cdot x^2) &= x^2y \cdot x^3 \xrightarrow[\text{property}]{\text{FLEX}_{(2,3)}} x^2(yx \cdot x^2) = x^2 \cdot yx^3 \Rightarrow \\ yx \cdot x^2 &= yx^3 \Rightarrow \text{RAP}^2 \text{ property.} \end{aligned}$$

Put $z = x^3$ in LTWC to get $x[(x \cdot yx)x^3] = (x \cdot xy)x^4$. Then,

$$\begin{aligned} \text{FLEX}_3 \text{ property in } G &\Rightarrow x[x \cdot (yx)x^3] = (x \cdot xy)x^4 \xrightarrow[\text{property}]{\text{LAP}^1} \\ x^2(yx \cdot x^3) &= x^2y \cdot x^4 \xrightarrow[\text{property}]{\text{FLEX}_{(2,4)}} x^2(yx \cdot x^3) = x^2 \cdot yx^4 \Rightarrow \\ yx \cdot x^3 &= yx^4 \Rightarrow \text{RAP}^3 \text{ property.} \end{aligned}$$

Put $z = x^n$ in LTWC to get $x[(x \cdot yx)x^n] = (x \cdot xy)x^{n+1}$. Then,

$$\begin{aligned} \text{FLEX}_n \text{ property in } G &\Rightarrow x[x \cdot (yx)x^n] = (x \cdot xy)x^{n+1} \xrightarrow[\text{property}]{\text{LAP}^1} \\ x^2(yx \cdot x^n) &= x^2y \cdot x^{n+1} \xrightarrow[\text{property}]{\text{FLEX}_{(2,n+1)}} x^2(yx \cdot x^n) = x^2 \cdot yx^{n+1} \\ &\Rightarrow yx \cdot x^n = yx^{n+1} \Rightarrow \text{RAP}^n \text{ property.} \end{aligned}$$

So, RAP^n property is true for all $n \in \mathbb{N}$.

Put $z = x^{-n}$ in LTWC to get $x[(x \cdot yx)x^{-n}] = (x \cdot xy)x^{-n+1}$. Then,

$$\begin{aligned} \text{FLEX}_{-n} \text{ property in } G &\Rightarrow x[x \cdot (yx)x^{-n}] = (x \cdot xy)x^{-n+1} \xrightarrow[\text{property}]{\text{LAP}^1} \\ x^2(yx \cdot x^{-n}) &= x^2y \cdot x^{-n+1} \xrightarrow[\text{property}]{\text{FLEX}_{(2,-n+1)}} x^2(yx \cdot x^{-n}) = x^2 \cdot yx^{-n+1} \Rightarrow \\ yx \cdot x^{-n} &= yx^{-n+1} \Rightarrow yx \cdot x^n = yx^{n+1} \Rightarrow \text{RAP}^{-n} \text{ property.} \end{aligned}$$

So, RAP^n property is true for all $n \in \mathbb{Z}^-$. Therefore, RAP^n property is true for all $n \in \mathbb{Z}$.

5. For any fixed $n \in \mathbb{Z}$, LPA^n -property hold if and only if $L_x^n = L_{x^n}$ for all $x \in G$. Thus, $L_x^{n+1} = L_x^n L_x = L_{x^n} L_x = L_x L_{x^n}$. So, LAP^n and LAP_n properties hold.
6. This follows by 5.
7. This is similar to 5.

□

3.2 Total inner mapping group of LTWC loop

We shall be discussing some properties of translations of LTWC-loop in this subsection for application in the next subsection.

Theorem 3.6. *In a LTWC loop $(G, \cdot)$ and for each $x, y \in G$:*

1. $R_x^m = L_x^{2n} R_x^m L_x^{-2n}$ or $[R_x^m, L_x^{2n}] = I$ for any $m, n \in \mathbb{Z}$.
2. $R_{x^n} = L_x^{-2} T_x^{n-2} (R_x L_x)^2 L_x^{n-2}$ for any $n \in \mathbb{Z}$.
3. $R_{x^n} = T_x^{n-1} R_x L_x^{n-1}$ for any $n \in \mathbb{Z}$.
4. $R_{(x,y)} = T_{(x,y)} L_x^{-1} S_x L_x = T_{(x,y)} T_x^{-1}$ where $S_x = R_x^{-1} L_x$ and $T_{(x,y)} = R_x R_y L_x^{-1} R_y^{-1}$ are elements of $\text{Inn}(G)$. $\text{Inn}(G) = \langle T_{(x,y)} \mid x, y \in G \rangle$ and $T\text{Inn}(G, \cdot) = \langle T_{(x,y)}, L_{(x,y)}, M_{(x,y)}, U_x \mid x, y \in G \rangle = \langle T_{(x,y)}, L_{(x,y)}, P_{(x,y)}, V_x \mid x, y \in G \rangle$.

Proof. G is a LTWC-loop if and only if for all $x, y \in G$,

$$R_x L_x R_z L_x = L_x^2 R_{xz} \quad (1)$$

A LTWC-loop G is an LCC-loop, and so, from the LCC-identity, for all $x, y \in G$,

$$R_y L_x = L_x R_x^{-1} R_{xy} \quad (2)$$

For all $x, y \in G$, the LCC-identity is equivalent to

$$L_x^{-1} L_y L_x = L_{xy/x} \quad (3)$$

Since G is a LTWC-loop, then $x(x^2 z) = x^2 \cdot xz$ for all $x, z \in G$. So, we have

$$L_x L_{x^2} = L_{x^2} L_x \quad \text{or} \quad [L_{x^2}, L_x] = I \quad (4)$$

1. Put $z = e$ in (1) to get $R_x L_x^2 = L_x^2 R_x$. So, $R_x = L_x^2 R_x L_x^{-2} \Rightarrow R_x = L_x^{2n} R_x L_x^{-2n}$ for all $n \in \mathbb{Z}$. Also, $R_x^2 = L_x^2 R_x^2 L_x^{-2}$, $R_x^3 = L_x^2 R_x^3 L_x^{-2}$, $\dots$, $R_x^m = L_x^2 R_x^m L_x^{-2}$. Then $R_x^m = L_x^{2n} R_x^m L_x^{-2n} \Leftrightarrow R_x^m L_x^{2n} = L_x^{2n} R_x^m \forall n, m \in \mathbb{Z} \Leftrightarrow [R_x^m, L_x^{2n}] = I \forall n, m \in \mathbb{Z}$.

2. Put $z = e$ in (1) to get $R_x L_x^2 = L_x^2 R_x \Rightarrow R_x = L_x^{-2} R_x L_x^2$
 $= L_x^{-2} (R_x L_x^{-1})^{-1} (R_x L_x)^2 L_x^{-1}.$

Put $z = x$ in (1), then $R_x L_x R_x L_x = L_x^2 R_{x^2} \Rightarrow R_{x^2} = L_x^{-2} R_x L_x R_x L_x =$

$$L_x^{-2} (R_x L_x^{-1})^0 (R_x L_x)^2 L_x^0.$$

Put $z = x^2$ in (1), then $R_x L_x R_{x^2} L_x = L_x^2 R_{x^3} \Rightarrow R_{x^3} = L_x^{-2} R_x L_x R_{x^2} L_x =$
 $L_x^{-2} R_x L_x L_x^{-2} R_x L_x R_x L_x L_x$

Put $z = x^3$ in (1) to get $R_x L_x R_{x^3} L_x = L_x^2 R_{x^4} \Rightarrow R_{x^4} = L_x^{-2} R_x L_x R_{x^3} L_x =$
 $L_x^{-2} R_x L_x L_x^{-2} R_x L_x^{-1} (R_x L_x)^2 L_x L_x = L_x^{-2} R_x L_x^{-1} R_x L_x^{-1} (R_x L_x)^2 L_x^2 =$
 $L_x^{-2} (R_x L_x^{-1})^2 (R_x L_x)^2 L_x^2.$

Put $z = x^4$ in (1), then $R_x L_x R_{x^4} L_x = L_x^2 R_{x^5} \Rightarrow R_{x^5} = L_x^{-2} R_x L_x R_{x^4} L_x =$
 $L_x^{-2} R_x L_x L_x^{-2} (R_x L_x^{-1})^2 (R_x L_x)^2 L_x^3 = L_x^{-2} (R_x L_x^{-1})^3 (R_x L_x)^2 L_x^3.$

Put $z = x^5$ in (1) to get $R_x L_x R_{x^5} L_x = L_x^2 R_{x^6} \Rightarrow R_{x^6} = L_x^{-2} R_x L_x R_{x^5} L_x$
 $= L_x^{-2} R_x L_x L_x^{-2} (R_x L_x^{-1})^3 (R_x L_x)^2 L_x^3 L_x = L_x^{-2} (R_x L_x^{-1})^4 (R_x L_x)^2 L_x^4.$

We claim that $R_{x^n} = L_x^{-2} (R_x L_x^{-1})^{n-2} (R_x L_x)^2 L_x^{n-2} \forall n \in \mathbb{N}$. When
 $n = 1, 2, 3, 4, 5, 6$ (true) going by computations above.

Let $R_{x^k} = L_x^{-2} (R_x L_x^{-1})^{k-2} (R_x L_x)^2 L_x^{k-2}$, for $n = k$.

Put $z = x^k$ in (1) to get $R_x L_x R_{x^k} L_x = L_x R_{x^{k+1}}.$

$$= L_x^{-2} R_x L_x L_x^{-2} (R_x L_x^{-1})^{k-2} (R_x L_x)^2 L_x^{k-2} L_x$$

$$= L_x^{-2} (R_x L_x^{-1})^{k-1} (R_x L_x)^2 L_x^{k-1}.$$

Replace $k + 1$ with n to get $R_{x^n} = L_x^{-2} (R_x L_x^{-1})^{n-2} (R_x L_x)^2 L_x^{n-2}.$

Put $z = x^{-1}$ in (1), then $R_x L_x R_{x^{-1}} L_x = L_x^2 R_e \Rightarrow R_{x^{-1}} = L_x^{-1} R_x^{-1} L_x^2 L_x^{-1}$
 $= L_x^{-1} R_x^{-1} L_x = L_x^{-2} (R_x L_x^{-1})^{-3} (R_x L_x)^2 L_x^{-3}.$

Put $z = x^{-2}$ in (1), then $R_x L_x R_{x^{-2}} L_x = L_x R_{x^{-1}} \Rightarrow R_{x^{-2}} = L_x^{-1} R_x^{-1} L_x^2 R_{x^{-1}} L_x^{-1}$
 $= L_x^{-1} R_x^{-1} L_x^2 (L_x^{-2} (R_x L_x^{-1})^{-3} (R_x L_x)^2 L_x^{-3}) L_x^{-1} = L_x^{-2} (R_x L_x^{-1})^{-4} (R_x L_x)^2 L_x^{-4}.$

We claim that $R_{x^n} = L_x^{-2} (R_x L_x^{-1})^{n-2} (R_x L_x)^2 L_x^{n-2} \forall n \in \mathbb{Z}^-$. When
 $n = -1, -2$ (true) going by computations above.

Assume that $R_{x^{-k}} = L_x^{-2} (R_x L_x^{-1})^{-k-2} (R_x L_x)^2 L_x^{-k-2}.$

Put $z = x^{-k-1}$ in (1) to get $R_x L_x R_{x^{-k-1}} L_x = L_x^2 R_{xx^{-k-1}} \Rightarrow R_x L_x R_{x^{-k-1}} L_x = L_x^2 R_{x^{-k}}$.

$$\begin{aligned} R_{x^{-(k+1)}} &= L_x^{-1} R_x^{-1} L_x^2 R_{x^{-k}} = L_x^{-1} R_x^{-1} L_x^2 (L_x^{-2} (R_x L_x^{-1})^{-k-2} (R_x L_x)^2 L_x^{-k-2} L_x^{-1}) \\ &= L_x^{-1} R_x^{-1} (R_x L_x^{-1})^{-k-2} (R_x L_x)^2 L_x^{-k-3} \\ &= L_x^{-2} (R_x L_x^{-1})^{-1} (R_x L_x^{-1})^{-k-2} (R_x L_x)^2 L_x^{-k-3} \\ &= L_x^{-2} (R_x L_x^{-1})^{-k-3} (R_x L_x)^2 L_x^{-k-3} \\ R_{x^{-(k+1)}} &= L_x^{-2} (R_x L_x^{-1})^{-(k+1)-2} (R_x L_x)^2 L_x^{-(k+1)-2}. \end{aligned}$$

Replace $k + 1$ with n to get $R_{x^{-n}} = L_x^{-2} (R_x L_x^{-1})^{-n-2} (R_x L_x)^2 L_x^{-n-2}$.

Therefore, $R_{x^n} = L_x^{-2} T_x^{n-2} (R_x L_x)^2 L_x^{n-2} \forall n \in \mathbb{Z}$.

3. Use (2) in the similar way in which (1) was used to prove 2.
4. Using (1), we get

$$\begin{aligned} R_{(x,y)} &= R_x R_y R_{xy}^{-1} = R_x R_y (L_x^{-2} R_x L_x R_y L_x)^{-1} = R_x R_y L_x^{-1} R_y^{-1} L_x^{-1} R_x^{-1} L_x^2 \\ &= T_{(x,y)} L_x^{-1} S_x L_x. \end{aligned}$$

Using (2), we get

$$\begin{aligned} R_{(x,y)} &= R_x R_y R_{xy}^{-1} = R_x R_y (R_x L_x^{-1} R_y L_x)^{-1} = R_x R_y L_x^{-1} R_y^{-1} L_x R_x^{-1} \\ &= T_{(x,y)} T_x^{-1}. \end{aligned}$$

□

Remark 3.7. By Theorem 3.6(4), LTWC-loop is thus a class of loop which falls into the Syrbu and Greco (2019) first answer to a question of Stanovský and Vojtěchovský (2014)-minimal generating set for the total inner mapping group of a loop. In addition, the class of LTWC-loop also provides an alternative solution to 2014 question and in fact, the inner automorphism group of a LTWC-loop is generated by just one inner mapping.

Corollary 3.8. Let G be a LTWC-loop. For any $x \in G$, $n \in \mathbb{N}$:

1. $|x| = n \Leftrightarrow T_x^n(R_x L_x)^2 L_x^n = T_x^2 L_x^4 \Leftrightarrow T_x^n L_x^n = I$.
2. $|T_x| = n \Leftrightarrow L_x^2 R_{x^{n+2}} = (R_x L_x)^2 L_x^n \Leftrightarrow R_{x^{n+1}} = R_x L_x^n$.
3. $|L_x| = n \Leftrightarrow L_x^2 R_{x^{n+2}} = T_x^n (R_x L_x)^2 \Leftrightarrow R_{x^{n+1}} = T_x^n R_x$.
4. G has the RAP^{n+1} -property if and only if $L_x^2 R_x R_{x^{n+1}} = T_x^n (R_x L_x)^2 L_x^n$; G has RAP if and only if $(x \cdot xy)x \cdot x = x \cdot (x \cdot yx)x$.
5. G has the RAP^n -property if and only if $R_x R_{x^n} = T_x^n R_x L_x^n$; G has RAP if and only if $T_x R_x = R_x T_x$.
6. G has the RAP_{n+1} -property if and only if $L_x^2 R_{x^{n+1}} R_x = T_x^n (R_x L_x)^2 L_x^n$.
7. G has the RAP_n -property if and only if $R_{x^n} R_x = T_x^n R_x L_x^n$.
8. G has the RPA^{n+2} -property if and only if $L_x^2 R_{x^{n+2}} = T_x^n (R_x L_x)^2 L_x^n$.
9. G has the RPA^{n+1} -property if and only if $R_{x^{n+1}} = T_x^n R_x L_x^n$.
10. G has the LPA^n -property if and only if $L_x^2 R_{x^{n+2}} = T_x^n (R_x L_x)^2 L_{x^n}$; G has LAP if and only if $(x \cdot xy)x^4 = x^2 \cdot x[x[x(x \setminus (yx) \cdot x) \cdot x] \cdot x]$.
11. G has the LPA^n -property if and only if $R_{x^{n+1}} = T_x^n R_x L_{x^n}$; G has LAP if and only if $yx^3 = x^2[x \setminus (yx) \cdot x]$.

Proof. We shall use Theorem 3.6.

1. Going by 2, $|x| = n \Leftrightarrow L_x^{-2} T_x^{n-2} (R_x L_x)^2 L_x^{n-2} = I \Leftrightarrow T_x^{n-2} (R_x L_x)^2 L_x^n = L_x^2 L_x^2 \Leftrightarrow T_x^n (R_x L_x)^2 L_x^n = T_x^2 L_x^4$. Also, going by 3, $|x| = n \Leftrightarrow T_x^{n-1} R_x L_x^{n-1} = I \Leftrightarrow T_x^{n-1} R_x L_x^{-1} L_x^n = I \Leftrightarrow T_x^n L_x^n = I$.
2. Similarly, use 2 and 3.
3. Similarly, use 2 and 3.

The proofs of 4 to 11 follow by using 2 and 3. □

Remark 3.9. In Corollary 3.8, we have established necessary and sufficient condition for a $LTWC$ -loop to have LAP (RAP) or to be left (right) power alternative.

Remark 3.10. *Going by Corollary 3.8(2), with $n = 1$, a LTWC-loop G is commutative if and only if $x \cdot yx = yx^2$ or $x[x \cdot (x \cdot yx)x] = (x \cdot xy)x^3$ for all $x, y \in G$.*

3.3 Construction of LTWC-loop

We shall be using some of the properties of the translations of LTWC-loop derived in previous subsection for construction. We adopt the following notations for such in a loop $(G, \cdot)$:

$$\prod_{\rho}(G, \cdot) = \prod_{\rho}(G) = \{R_a \mid a \in G\} = \{R(a) \mid a \in G\} \quad \text{and}$$

$$\prod_{\lambda}(G, \cdot) = \prod_{\lambda}(G) = \{L_a \mid a \in G\} = \{L(a) \mid a \in G\}.$$

Designate the elements of G by $x_1, \dots, x_n$ and R_{x_j} by $R_j = R(j)$, and assume henceforth that R_1 is the identity permutation. Then $R_2, \dots, R_n$, are permutations on n letters and $R(i) = (1 \ i \ \dots)$. Conversely if $(1 \ i \ \dots)$ is in Π_{ρ} , then $(1 \ i \ \dots) = R_i = R(i)$.

Theorem 3.11. *Let G be a loop.*

1. *G is a LTWC-loop if and only if for all $\alpha \in \prod_{\rho}$, for some $\beta \in \prod_{\lambda}$, and for some $\alpha' \in \prod_{\rho}$, $\beta^{-2}\alpha'\beta\alpha\beta \in \prod_{\rho}$.*
2. *If G is a LTWC-loop, then*
 - (a) $\beta, \beta' \in \prod_{\lambda} \Rightarrow \beta^{-1}\beta'\beta \in \prod_{\lambda}$.
 - (b) for some $\alpha \in \prod_{\lambda}$ and $\beta \in \prod_{\rho}$, $\beta\alpha^2 = \alpha^2\beta$.
 - (c) for some $\alpha \in \prod_{\lambda}$ and $\beta \in \prod_{\rho}$, $\alpha^{-2}(\beta\alpha^{-1})^{n-2}(\beta\alpha)^2\alpha^{n-2} \in \prod_{\rho}$ for all $n \in \mathbb{Z}$.
 - (d) for some $\alpha \in \prod_{\lambda}$ and $\beta \in \prod_{\rho}$, $(\beta\alpha^{-1})^{n-1}\beta\alpha^{n-1} \in \prod_{\rho}$ for all $n \in \mathbb{Z}$.
 - (e) for some $\alpha \in \prod_{\lambda}$ and $\beta \in \prod_{\rho}$, $\beta^m\alpha^{2n} = \alpha^{2n}\beta^m \ \forall \ m, n \in \mathbb{Z}$.
 - (f) for some $\beta, \beta' \in \prod_{\lambda}$, $\beta\beta' = \beta'\beta$.

(g) for all $\alpha \in \prod_\rho$, for some $\beta \in \prod_\lambda$, and for some $\alpha' \in \prod_\rho$, $\alpha' \beta^{-1} \alpha \beta \in \prod_\lambda$.

Proof. Use Theorem 3.6, including (1), (2), (3) and (4). $\square$

Theorem 3.12. *Let G be a LTWC-loop.*

1. *If there exists $x \in G$ such that $G = \langle x \rangle$, then for some $\beta \in \prod_\rho$,*

$$\prod_\rho(G) = \left\{ \alpha^{-2} (\beta \alpha^{-1})^{n-2} (\beta \alpha)^2 \alpha^{n-2} \mid \text{for some } \alpha \in \prod_\lambda, n \in \mathbb{Z} \right\}$$

and G is a group.

2. *If there exist $\{x_i\}_{i=1}^{r \geq 2} \subset G$ such that $\bigcup_{i=1}^r \langle x_i \rangle = G$, then for some $\beta \in \prod_\rho$,*

$$\prod_\rho(G) = \left\{ \alpha^{-2} (\beta \alpha^{-1})^{n-2} (\beta \alpha)^2 \alpha^{n-2} \mid \text{for some } \alpha \in \prod_\lambda, n \in \mathbb{Z} \right\}.$$

3. *If there exists $x \in G$ such that $G = \langle x \rangle$, then for some $\beta \in \prod_\rho$,*

$$\prod_\rho(G) = \left\{ (\beta \alpha^{-1})^{n-1} \beta \alpha^{n-1} \mid \text{for some } \alpha \in \prod_\lambda, n \in \mathbb{Z} \right\}$$

and G is a group.

4. *If there exist $\{x_i\}_{i=1}^{r \geq 2} \subset G$ such that $\bigcup_{i=1}^r \langle x_i \rangle = G$, then for some $\beta \in \prod_\rho$,*

$$\prod_\rho(G) = \left\{ (\beta \alpha^{-1})^{n-1} \beta \alpha^{n-1} \mid \text{for some } \alpha \in \prod_\lambda, n \in \mathbb{Z} \right\}.$$

5. If there exist $\{x_i\}_{i=1}^{r \geq 2} \subset G$ such that $\bigcup_{i,j=1}^r \langle x_i, x_j \rangle = G$, then given any $\alpha \in \prod_\rho$ and for some $\alpha' \in \prod_\rho$, $\prod_\rho(G) = \left\{ \beta^{-2} \alpha' \beta \alpha \beta \mid \text{for some } \beta \in \prod_\lambda \right\}$.
6. If there exist $\{x_i\}_{i=1}^{r \geq 2} \subset G$ such that $\bigcup_{i,j=1}^r \langle x_i, x_j \rangle = G$, then given any $\alpha \in \prod_\rho$ and for some $\alpha' \in \prod_\rho$, $\prod_\rho(G) = \left\{ \alpha' \beta^{-1} \alpha \beta \mid \text{for some } \beta \in \prod_\lambda \right\}$.

Proof. Let G be a LTWC-loop. We shall use Theorem 3.6 and Theorem 3.11.

1. If there exists $x \in G$ such that $G = \langle x \rangle$, then $R_{x^n} = L_x^{-2} T_x^{n-2} (R_x L_x)^2 L_x^{n-2}$ for any $n \in \mathbb{Z}$ implies $\prod_\rho(G) = \left\{ \alpha^{-2} (\beta \alpha^{-1})^{n-2} (\beta \alpha)^2 \alpha^{n-2} \mid \text{for some } \alpha \in \prod_\lambda, n \in \mathbb{Z} \right\}$ for some $\beta \in \prod_\rho$. A LTWC-loop is power associative, thus, $G = \langle x \rangle$ is a group.
2. If there exist $\{x_i\}_{i=1}^{r \geq 2} \subset G$ such that $\bigcup_{i=1}^r \langle x_i \rangle = G$, then $R_{x^n} = L_x^{-2} T_x^{n-2} (R_x L_x)^2 L_x^{n-2}$ for any $n \in \mathbb{Z}$ implies $\prod_\rho(G) = \left\{ \alpha^{-2} (\beta \alpha^{-1})^{n-2} (\beta \alpha)^2 \alpha^{n-2} \mid \text{for some } \alpha \in \prod_\lambda, n \in \mathbb{Z} \right\}$ for some $\beta \in \prod_\rho$.
3. This is similar to 1 using $R_{x^n} = T_x^{n-1} R_x L_x^{n-1}$ for any $n \in \mathbb{Z}$ in a LTWC-loop.
4. This is similar to 2 using $R_{x^n} = T_x^{n-1} R_x L_x^{n-1}$ for any $n \in \mathbb{Z}$ in a LTWC-loop.
5. This is similar to 2 using $R_{xy} = L_x^{-2} R_x L_x R_y L_x$ with $x, y \in \{x_i\}_{i=1}^{r \geq 2}$ in a LTWC-loop.

6. This is similar to 2 using $R_{xy} = R_x L_x^{-1} R_y L_x$ with $x, y \in \{x_i\}_{i=1}^{r \geq 2}$ in a LTWC-loop.

□

Example 3.13. Let G be a LTWC-loop of order 8. Given

$$\begin{aligned} \beta = \beta_7 &= (1 \ 6 \ 4 \ 7)(2 \ 8)(3 \ 5) = R(6), \quad \beta' = \beta_8 = (1 \ 2)(3 \ 4)(5 \ 7 \ 8 \ 6) = R(2), \\ \alpha &= (1 \ 6 \ 4 \ 7)(2 \ 5 \ 3 \ 8) = L(6), \quad \alpha' = (1 \ 2)(3 \ 4)(5 \ 7)(6 \ 8) = L(2) \in \prod_{\lambda}(G). \end{aligned}$$

We shall now apply Theorem 3.11.

$$\text{When, } n = 2, \quad \beta_1 = (\beta\alpha^{-1})^{2-1} \beta\alpha^{2-1} = (1 \ 4)(2 \ 3)(5 \ 8)(6 \ 7) = R(4) \in \prod_{\rho}(G).$$

$$\text{When, } n = 3, \quad \beta_2 = (\beta\alpha^{-1})^{3-1} \beta\alpha^{3-1} = (1 \ 7 \ 4 \ 6)(2 \ 5)(3 \ 8) = R(7) \in \prod_{\rho}(G).$$

$$\text{When, } n = 4, \quad \beta_3 = (\beta\alpha^{-1})^{4-1} \beta\alpha^{4-1} = (1)(2)(3)(4)(5)(6)(7)(8) = R(1) \in \prod_{\rho}(G).$$

$$\beta_4 = \beta\alpha^{-1}\beta'\alpha = (1 \ 5)(2 \ 7)(3 \ 6)(4 \ 8) = R(5) \in \prod_{\rho}(G).$$

$$\beta_5 = \beta'\alpha'^{-1}\beta\alpha' = (1 \ 8)(2 \ 6)(3 \ 7)(4 \ 5) = R(8) \in \prod_{\rho}(G).$$

$$\beta_6 = \beta'\alpha'^{-1}\beta_1\alpha' = (1 \ 3)(2 \ 4)(5 \ 6 \ 8 \ 7) = R(3) \in \prod_{\rho}(G).$$

Then, $\prod_{\rho}(G) = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8\}$. Consequently, the multiplication table of G is presented in Table 1.

4 Concluding Remarks

The generalized RAP, LAP and FLEX properties in Definition 3.1 can be viewed as the 'measure' of difference between a LTWC (RTWC)-loop being power associative and not being diassociative viz-a-vis the results in Theorem 3.5 and Corollary 3.8.

$\cdot$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	7	8	5	6
3	3	4	1	2	6	5	8	7
4	4	3	2	1	8	7	6	5
5	5	7	6	8	1	3	2	4
6	6	5	8	7	3	4	1	2
7	7	8	5	6	2	1	4	3
8	8	6	7	5	4	2	3	1

Table 1: A LTWC-loop $(G, \cdot)$ of order 8

In literature, there are some varieties of loops of first Bol-Moufang types which have RAP, LAP, power right (left) alternative properties (e.g. LC, RC, central [1, 9], left/right Bol) which are power associative but not necessarily diassociative. There are also some varieties of loops first Bol-Moufang type which have RAP, LAP, power right (left) alternative properties, flexibility (e.g. Moufang, extra) which are both power associative and diassociative. What can then be said about varieties of loops of second Bol-Moufang types such as LTWC (RTWC)-loop ?

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